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Topological charges in field theory

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Abstract. Non-dynamical conserved quantities are analysed for a class of field theories and their topological character is discussed. Some examples are given.

1. Introduction

Classical solutions of field theories have received much attention recently (Goldstone and Jackiw 1975, Dashen *et al* 1974, Faddeev 1975). The interest in these solutions derives from the fact that they exhibit extended particle structures (kinks, solitons). They also offer a new and potentially powerful approach to the bound state problem in quantum field theory.

In many models a conserved quantity is obtained and this has the interesting property that its appearance is not due to any particular symmetry in the theory nor due to any particular form of dynamics. Rather its appearance depends on the manifold on which the field variables take their values and hence it is called a topological charge.

In this paper we should like to study these charges in a systematic way for a class of field theories. Most of the results we present are known in one form or another in the literature (Faddeev 1975). Our presentation however does make clear the nature of these conserved charges. We use the methods of differential topology (Milnor 1965, Guillemin and Pollack 1974) which were so elegantly employed by Arafune *et al* (1975) in their analysis of the conserved magnetic charge discovered by t'Hooft (1974). The relationship of our analysis with that of various authors (Faddeev 1975, Finkelstein 1966) who use homotopy groups is discussed. In § 2 we give a detailed presentation of the conserved quantities and exhibit their topological character and in § 3 we consider some examples.

2. Topological charges

2.1. Non-dynamical conservation laws

We begin our analysis with the following simple theorem.

Theorem. Let $\hat{\phi} = (\hat{\phi}^1, \ldots \hat{\phi}^n)$ be an n-dimensional smooth, unit vector field‡ defined

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‡ For a smooth unit vector field, all partial derivatives exist and are continuous and $(\hat{\phi}, \hat{\phi}) = (\hat{\phi}^1)^2 + \ldots + (\hat{\phi}^n)^2 = 1$.

on a d-dimensional space–time. Then, if $n \leq d$, non-dynamical conserved quantities exist defined in terms of these fields.

Proof. Since $(\hat{\phi}, \hat{\phi}) = 1$, $(\hat{\phi}, \partial_\mu \hat{\phi}) = 0$. Thus if $n \leq d$, the $d \times n$ matrix $[\partial \hat{\phi}]$ must have a rank smaller than n, i.e.:

$$\epsilon^{\mu_1 \ldots \mu_n \ldots \mu_d} \epsilon_{\alpha_1 \ldots \alpha_n} \partial_{\mu_1} \hat{\phi}^{\alpha_1} \ldots \partial_{\mu_n} \hat{\phi}^{\alpha_n} = 0 \tag{2.1}$$

or

$$\partial_{\mu_1} [\epsilon^{\mu_1 \ldots \mu_n \ldots \mu_d} \epsilon_{\alpha_1 \ldots \alpha_n} \hat{\phi}^{\alpha_1} \ldots \partial_{\mu_n} \hat{\phi}^{\alpha_n}] = 0. \tag{2.2}$$

The quantity in the square bracket in (2.2) is therefore conserved. It is clearly non-dynamical; rather it is geometric.

The theorem tells us how we may construct conserved quantities for theories like the non-linear σ model. In many other theories one has an n-dimensional vector field ϕ which is not of unit length. We may then consider the direction vector field $\hat{\phi} = \phi / \|\phi\|$. However this is not smooth everywhere and expression (2.1) constructed from it need not vanish everywhere. In fact in this case $\hat{\phi}$ is smooth except at the zeros of ϕ (assuming ϕ is smooth) and we expect that the left-hand side of (2.1) shows δ-function singularities at the zeros of ϕ. For these theories however one would still have (if $n+1 \leq d$):

$$\partial_{\mu_{n+1}} [\epsilon^{\mu_1 \ldots \mu_n \ldots \mu_d} \epsilon_{\alpha_1 \ldots \alpha_n} \partial_{\mu_1} \hat{\phi}^{\alpha_1} \ldots \partial_{\mu_n} \hat{\phi}^{\alpha_n}] = 0. \tag{2.3}$$

The quantity in the square bracket is then conserved.

For further analysis we restrict ourselves to the following two cases:

(A) $n = d$, $\hat{\phi}$ is smooth everywhere.
(B) $n + 1 = d$, $\hat{\phi} = \phi / \|\phi\|$, ϕ is smooth everywhere.

Most of the known models in which one meets topological charges fall into these two cases.

Following (2.2), for case A, we define the conserved current by:

$$\mathcal{J}^\mu_{\mathrm{A}} = \frac{1}{(n-1)!\Omega^{n-1}} \epsilon^{\mu \mu_2 \ldots \mu_n} \epsilon_{\alpha_1 \ldots \alpha_n} \hat{\phi}^{\alpha_1} \partial_{\mu_2} \hat{\phi}^{\alpha_2} \ldots \partial_{\mu_n} \hat{\phi}^{\alpha_n} \tag{2.4}$$

and following (2.3), for case B, we define

$$\mathcal{J}^\mu_{\mathrm{B}} = \frac{1}{(n-1)!\Omega^{n-1}} \epsilon^{\mu \mu_1 \ldots \mu_n} \epsilon_{\alpha_1 \ldots \alpha_n} \partial_{\mu_1} \hat{\phi}^{\alpha_1} \ldots \partial_{\mu_n} \hat{\phi}^{\alpha_n} \tag{2.5}$$

as the conserved current. In (2.4) and (2.5), Ω^{n-1} corresponds to the surface area of a unit sphere in n-dimensional space.

From our discussion following (2.2) we should notice that $\mathcal{J}^\mu_{\mathrm{A}}$ is a smooth current distribution. On the other hand $\mathcal{J}^\mu_{\mathrm{B}}$ is zero everywhere except at the zeros of $\phi(x)$. We also notice that in spite of being conserved, these currents do not generate any symmetry transformations since the charge density contains no time derivatives of the field variables and hence commutes with them.

The charge in volume V of space in the two cases is given by:

$$Q^V_{\mathrm{A}} = \int_V \mathrm{d}x^{n-1} \mathcal{J}^0_{\mathrm{A}}(x) = \frac{1}{\Omega^{n-1}} \int_V \mathrm{d}x^{n-1} \sqrt{\det g_{ab}} \left\| \frac{\partial \xi}{\partial x} \right\| \tag{2.6}$$

and

$$Q_{\mathrm{B}}^{V} = \int_{V} \mathrm{d}x^{n} \mathscr{J}_{\mathrm{B}}^{0}(x) = \frac{1}{\Omega^{n-1}} \int_{\partial V} \mathrm{d}y^{n-1} \sqrt{\det g_{ab}} \left\| \frac{\partial \xi}{\partial y} \right\| \tag{2.7}$$

where $\xi^{a} = (\xi^{1}, \ldots, \xi^{n-1})$ are the intrinsic coordinates on the sphere $S_{\hat{\phi}}^{n-1}$, $g_{ab} = (\partial \hat{\phi}/\partial \xi^{a}, \partial \hat{\phi}/\partial \xi^{b})$ is the metric on $S_{\hat{\phi}}^{n-1}$ and, in (2.7) $(y^{1}, \ldots, y^{n-1})$ are the intrinsic coordinates on the surface ∂V of V.

2.2. *Boundary conditions*

In order to evaluate these integrals for solutions of physical interest we should consider further conditions on $\hat{\phi}(x)$. We shall restrict ourselves to the following types of boundary conditions:

Case A. For the convergence of the energy integral we require that

$$\hat{\phi}(\boldsymbol{x}) \rightarrow \hat{\phi}_{\mathrm{vac}} \qquad \text{as} \qquad \|\boldsymbol{x}\| \rightarrow \infty$$

where $\hat{\phi}_{\mathrm{vac}}$ is one particular vacuum solution.

Case B. Here we require a somewhat different condition. In order to avoid infinite charge solutions we demand that the vacuum solution $\phi_{\mathrm{vac}} \neq 0$. This is always so for a theory with a spontaneously broken symmetry. For the energy to be finite this condition would mean that ϕ can be zero only in a bounded region. We shall assume that ϕ has a finite number of isolated zeros.

2.3. *Degree of a map*

Let us now introduced the topological concept of the degree of a map. This will enable us to solve the integrals for the charges. The degree of a map from a manifold M to another manifold N is effectively the number of times N is covered as one varies over M. The conditions that M and N must satisfy for the concept to be meaningful are specified in the following formal definition (Milnor 1965, Guillemin and Pollack 1974).

Let M and N be two differentiable manifolds of dimension n, M being compact and N connected. Let f be a smooth mapping from M to N. Let $M_{y} = \{x \in M\colon f(x) = y, y \in N\}$, that is, the set of points of N which take the value y under f. Choose y to be the regular value of f (i.e. the Jacobian $\|\partial f/\partial x\|$ is non-zero at all points in M_{y}). Then the sum of the sign of the Jacobian $\|\partial f/\partial x\|$ at all the points of M_{y} is called the *degree of the map* f:

$$\deg(f) := \sum_{x \in M_{y}} \operatorname{sgn} \left\| \frac{\partial f}{\partial x} \right\|. \tag{2.8}$$

This definition is independent of y (because of connectedness) and is globally defined since regular values of f are dense in $f(M)$. Thus $\deg(f)$ is just a count, taking orientation into account, of the points which are mapped into a regular value.

We evaluate the integral for the charges by the use of the degree theorem (Guillemin and Pollack 1974):

Let (x^{i}) be the coordinates on M and (ξ^{i}) the coordinates on N and further, let N be also compact. Given a real valued function h on N, it defines a real valued function $h_{0}f$ on M and we have:

$$\int_{M} \mathrm{d}x^{n} (h_{0}f)(x) \left\| \frac{\partial \xi}{\partial x} \right\| = \deg(f) \int_{N} \mathrm{d}\xi^{n} h(\xi). \tag{2.9}$$

This formula is easy to understand. The integral of h over N is carried out deg(f) times as x varies over M.

2.4. Evaluation of the charges

We now use these concepts and the specified boundary conditions to evaluate the integrals (2.6) and (2.7). Notice that in both cases $N = S_{\hat{\phi}}^{n-1}$ and

$$\int_N \mathrm{d}\xi^{n-1}\sqrt{\det g_{ab}} = \Omega^{n-1}. \tag{2.10}$$

Consider first the integral (2.6). The charge Q_{A}^{V}, for arbitrary V, is some real number. To determine the total charge $Q_{\mathrm{A}}^{\mathrm{tot}}$, we may use the boundary conditions to compactify $\mathbb{R}^{n-1}$ by some map

$$c: \mathbb{R}^{n-1} \cup \{\infty\} \to S^{n-1}.$$

Then it is easy to see that:

$$Q_{\mathrm{A}}^{\mathrm{tot}} = \deg(\hat{\phi}_0 c^{-1}).$$

The total charge is therefore an integer.

Consider now the integral (2.7). Since we have assumed that ϕ has a finite number of isolated zeros, we may take ∂V to be the surface of a sphere of some radius R so that ϕ has no zeros on ∂V. The map $\hat{\phi}$ restricted to ∂V is smooth and using the degree theorem we obtain:

$$Q_{\mathrm{B}}^{\mathrm{tot}} = \deg(\hat{\phi}|_{\partial V}).$$

Thus the charge in case B is always localized and an integer which is in sharp contrast to case A.

It is an interesting question to ask how many values the total topological charge can take. In the cases that we have considered the charge equals the degree of a certain map and this can take any integer value.

What does the charge characterize? Since it is of topological origin we expect that it is invariant under continuous deformations of the field $\hat{\phi}$. This is indeed so. All maps from one topological space to another can be divided into the so-called homotopy classes; two maps belong to the same class if they can be continuously deformed into each other. For mappings of S^n to S^m (which is of interest to us) these classes are in one-to-one correspondence with the elements of the homotopy groups $\Pi_n(S^m)$ (Finkelstein 1966). For $n = m$ the concept of degree is defined and a theorem due to Hopf states that two maps belong to the same homotopy class if and only if they have the same degree. The degree of $\hat{\phi}$ thus characterizes the homotopy class to which the field $\hat{\phi}$ belongs.

2.5. Conserved tensor densities

For cases besides A and B, we may define conserved tensor densities and it is interesting to ask how one may extract a topological charge from them. In the case $n < d$ and $\hat{\phi}$ smooth, one has from (2.2)

$$\mathscr{J}^{\mu_n\mu_{n+1}\cdots\mu_d} = \epsilon^{\mu_1\cdots\mu_n\cdots\mu_d}\epsilon_{\alpha_1\ldots\alpha_n}(\partial_{\mu_1}\hat{\phi}^{\alpha_1})\ldots(\partial_{\mu_{n-1}}\hat{\phi}^{\alpha_{n-1}})\hat{\phi}^{\alpha_n} \tag{2.11}$$

which is conserved.

It is easily seen that charges constructed from this tensor density by integrating over various surfaces are constants of motion and surface independent. They can all be related as appropriate integrals of

$$Q(t_0, x_0^{n+1}, \ldots, x_0^d) = \int_{\substack{t=t_0 \\ x^{n+1}=x_0^{n+1} \\ \vdots \\ x^d = x_0^d}} d\sigma_{\mu_n \ldots \mu_d} \mathcal{J}^{\mu_n \cdots \mu_d}(x) \tag{2.12}$$

which itself is conserved and surface independent (i.e. independent of t_0 and x_0^i, $i = n+1, \ldots, d$). This integral is clearly zero since it may be evaluated at a surface at infinity where the current density vanishes. Thus all charges defined as surface or volume integrals of the tensor current densities are identically zero and contain no topological information. One may be tempted to say that the topological charge must be zero. This however is incorrect since the topological charge, as we define it, must determine the homotopy class of the field and therefore must be an element of the appropriate homotopy group. There are examples where the above charges are zero but where the associated homotopy groups are not trivial. This happens, for example, in the O(3) symmetric non-linear σ model in four space–time dimensions or, as may be explicitly verified, for the map $\hat{\phi} = \psi_0 c$ where c is the stereographic projection: $\mathbb{R}^3 \cup \{\infty\} \to S^3$ and ψ is the map:

$$(\sin \tfrac{1}{2}\alpha \cos \beta, \sin \tfrac{1}{2}\alpha \sin \beta, \cos \tfrac{1}{2}\alpha \cos \gamma, \cos \tfrac{1}{2}\alpha \sin \gamma)$$
$$\to (\sin \alpha \cos(\beta - \gamma), \sin \alpha \sin(\beta - \gamma), \cos \alpha)$$

for which the topological invariant is called the Hopf invariant and equals one but for which the charge Q in (2.12) is zero. We expect that these conserved current densities should contain topological information but how one may determine the topological charge from them is at present not clear.

3. Examples

In this section we consider various models which fall into cases A or B.

3.1. *Case* A

3.1.1. *Sine–Gordon equation.*

$$\Box\varphi + \sin \varphi = 0, \qquad d = 2. \tag{3.1}$$

Since the equations of motion are invariant under $\varphi \to \varphi + 2\pi$ we can embed the theory in a non-linear σ model by defining:

$$\hat{\phi}^1 = \cos \varphi, \qquad \hat{\phi}^2 = \sin \varphi. \tag{3.2}$$

We are interested in smooth solutions for φ and therefore $\hat{\phi}^1$ and $\hat{\phi}^2$ are also smooth everywhere.

The topological current is defined to be:

$$\mathcal{J}^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu} \epsilon_{ab} \hat{\phi}^a \partial_\nu \hat{\phi}^b \tag{3.3}$$

and we notice that it is smooth and therefore a continuous distribution. The charge in this case is easily evaluated and, in the range $[a, b]$, is:

$$Q^{[a,b]} = \frac{1}{2\pi}[\varphi(b) - \varphi(a)]$$

which is, in general, not an integer. The total charge is given by:

$$Q^{\text{tot}} = \frac{1}{2\pi}[\varphi(\infty) - \varphi(-\infty)]. \tag{3.4}$$

The boundary condition in this case is $\varphi \to$ a vacuum solution as $x \to \pm\infty$ and since the vacuum solutions differ by $2\pi n$ (n integer), it follows that:

$$Q^{\text{tot}} = n.$$

The same result would of course be obtained by considerations of § 1.

3.1.2. Non-linear σ models.

$$\Box\hat{\phi} + \hat{\phi}(\partial_\mu\hat{\phi}, \partial^\mu\hat{\phi}) = 0, \qquad (\hat{\phi}, \hat{\phi}) = 1. \tag{3.5}$$

For the case $n = d = 3$ an infinite number of kink solutions were presented by Honerkamp *et al* (1976). For this case $\hat{\phi}$ is smooth everywhere and the topological current is given by:

$$\mathcal{J}^\mu = \frac{1}{8\pi}\epsilon^{\mu\nu\sigma}\epsilon_{abc}\hat{\phi}^a\partial_\nu\hat{\phi}^b\partial_\sigma\hat{\phi}^c. \tag{3.6}$$

For $n = d = 4$ we have the non-linear chiral Lagrangian for pions. The topological charge is defined from the current†:

$$\mathcal{J}^\mu = \frac{1}{3!4\pi^2}\epsilon^{\mu\nu\sigma\rho}\epsilon_{abcd}\hat{\phi}^a\partial_\nu\hat{\phi}^b\partial_\sigma\hat{\phi}^c\partial_\rho\hat{\phi}^d. \tag{3.7}$$

The currents (3.6) and (3.7) are continuous distributions and the total charge is an integer if the prescribed boundary conditions are used.

3.2. *Case* B

3.2.1. Self-coupled scalar fields. A neutral, self-coupled scalar field ($n + 1 = d = 2$, ϕ smooth):

$$\Box\phi + \frac{\partial V}{\partial\phi} = 0.$$

We define: $\hat{\phi}(x) = \phi(x)/|\phi(x)|$ whenever $\phi(x) \neq 0$. The topological current in this case is:

$$\mathcal{J}^\mu = \epsilon^{\mu\nu}\partial_\nu\hat{\phi} \tag{3.8}$$

and if $x^1, \ldots, x^n$ are the zeros of $\phi(x)$ then the current density is:

$$\mathcal{J}^0(x) = \frac{\partial\hat{\phi}(x)}{\partial x} = \sum d_{\hat{\phi}}(x^i)\delta(x - x^i) \tag{3.9}$$

† For a kink solution in a model belonging to this class, see Skyrme (1961).

where $d_{\hat{\phi}}(x^i) = \pm 1, 0$ depending on the behaviour of $\phi(x)$ near x^i†. It is clear that the charge is localized and is *always* an integer and, from continuity of $\phi(x)$, if follows that the total charge is ± 1 or 0.

3.2.2. t'Hooft's monopole (Arafune *et al* 1975, t'Hooft 1974). Here the magnetic current density is $(n+1=d=4)$:

$$\mathcal{J}_g^{\mu} = \frac{1}{2e}\epsilon^{\mu\nu\sigma\rho}\epsilon_{abc}\partial_\nu\hat{\phi}^a\,\partial_\sigma\hat{\phi}^b\,\partial_\rho\hat{\phi}^c \qquad (3.10)$$

which differs by a factor of $4\pi/e$ from the topological current (2.5). Magnetic charge is therefore localized and always quantized in the units of $4\pi/e$.

4. Conclusion

We have shown how conserved quantities of topological character can appear in field theory. In our analysis they arise because of constraints on the field variables. Such constraints are, of course, already present in theories containing unit vector fields. For other theories one may either consider vector fields or suitably embed the theory in higher dimensions as we did for the Sine–Gordon equation. We have considered two classes of models in detail (cases A and B in the text). In case A the topological current is a continuous distribution and the charge in any volume is generally an arbitrary real number. With the prescribed boundary condition the total charge is shown to be an integer specifying the homotopy class of the field variable. In case B, with suitable boundary condition the charge is always localized and always an integer.

Most of the known models where topological charges have been explicitly displayed belong to class A or B that we have considered. There are however models where one has conserved topological tensor currents and for which one expects non-trivial topological invariants. So far it remains unclear how these invariants may be extracted from the currents.

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† For the well known static kink solution in ϕ^4 theory and the two-soliton solution of Christ and Lee (1975):

$$\hat{\phi}(x) = \theta(x) \qquad d_{\hat{\phi}}(0) = 1.$$

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